Finite size corrections to scaling in high Reynolds number turbulence

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We study analytically and numerically the corrections to scaling in turbulence which arise due to the finite ratio of the outer scale L of turbulence to the viscous scale η, i.e., they are due to finite size effects as anisotropic forcing or boundary conditions at large scales. We find that the deviations $\delta\zeta_m$ from the classical Kolmogorov scaling $\zeta_m = m/3$ of the velocity moments $\langle|\boldsymbol{u}(\boldsymbol{k})|^m\rangle \propto k^{-\zeta_m}$ decrease like $\delta\zeta_m(Re) = c_m Re^{-3/10}$. Our numerics employ a reduced wave vector set approximation for which the small scale structures are not fully resolved. Within this approximation we do not find Re independent anomalous scaling within the inertial subrange. If anomalous scaling in the inertial subrange can be verified in the large Re limit, this supports the suggestion that small scale structures should be responsible, originating from viscosity either in the bulk (vortex tubes or sheets) or from the boundary layers (plumes or swirls).

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A major question in the theory of turbulent flows is whether there exists an asymptotic scaling state at high Re, and whether in this state the values of the scaling exponents conform with the classical predictions of the Kolmogorov theory [1]. There is some experimental evidence that a scaling state exists, and that the scaling exponents deviate from the Kolmogorov prediction (anomalous scaling). Theoretically there is still no proof for anomalous scaling based on the Navier-Stokes equations. Recently it was suggested that the reason for deviations from the Kolmogorov theory is related to the creation of small scale structures like vorticity sheets and tubes in which the local geometry is not 3-dimensional and isotropic [2].

One could hope that numerical simulations could be used to decide whether classical or anomalous scaling should be expected in high Re flows. If anomalous scaling were found, simulations could distinguish the responsible physical mechanism. Unfortunately, in direct simulations of the Navier-Stokes equations with current computers one can achieve only up to Taylor-Reynolds numbers of about $Re_\lambda = 200$ [3]. The range of scales for which power law behavior is observed at such values of Re_λ is too small to distinguish between classical and anomalous scaling. In addition, at low to moderate values of Re one can have corrections to scaling due to finite size effects, and it is hard to take these into account if their expected Re dependence is not known. It is this latter issue which is the focus of this letter. We examine these corrections to scaling analytically and numerically, and show how to take them into account in any future experiment or numerical simulation.

A high Re flow can be constructed in a numerical simulation by using a reduced wave vector set approximation (Fourier-Weierstrass decomposition), that was introduced and studied extensively by two of us recently. For a detailed description and references of previous work we refer to refs. [4, 5, 6]. The main idea is to start with a regular Fourier decomposition of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ in terms of plane waves $\exp(i\boldsymbol{k}\cdot\boldsymbol{x})$, but admit only a geometrically scaling subset $K = \cup_l K_l$ of wave vectors in the Fourier sum,

$$\boldsymbol{u}(\boldsymbol{x}, t) = \sum_{k\in K} \boldsymbol{u}(\boldsymbol{k}, t) \exp(i\boldsymbol{k}\cdot\boldsymbol{x}). \quad (1)$$

The subset of wave vectors is chosen such that $K_0 = \{k_n^{(0)}, n = 1, \ldots, N\}$ and $K_l = \{2^l k_n^{(0)}, n = 1, \ldots, N\}$, $l = 1, \ldots, l_{max}$. The basic set K_0 is chosen to contain wave vectors of different lengths that interact dynamically to a good degree, and l_{max} is chosen large enough to guarantee that the amplitudes $\boldsymbol{u}(\boldsymbol{k}_n^{(l_{max})}, t)$ of the smallest scales are practically zero. This of course depends on the viscosity ν and the Reynolds number Re. The flow is driven by a deterministic, non stochastic driving $\boldsymbol{f}(\boldsymbol{k}, t)$ with $\boldsymbol{k}$ in K_0. In fact, only the seven smallest wave vectors in K_0 were driven. Obviously, this driving is not isotropic. The k-range of externally forced amplitudes is denoted henceforth as stirring subrange.

The simulations that we refer to here used a value of $N = 80$ and scanned a range of Re from 10^4 to 10^7 [6]. It should be stressed that in this scheme the density of wave vectors per k interval decreases like $1/k$, whereas it increases like k^2 in full grid simulations. For this reason small scale structures (as vortex tubes or sheets)

cannot be resolved very accurately. But on the other hand many more scales than in full simulations can be examined. For the largest Re our simulations comprise more than three orders of magnitude between the wave number with maximum dissipation and the outer scale. The scaling exponents ζ_m can be determined with high accuracy.

The scaling exponents were measured in this numerical simulation by computing the m^{th}-order spectra $\langle|\boldsymbol{u}(\boldsymbol{k})|^m\rangle$, and fitting the three parameter function

$$\langle|\boldsymbol{u}(\boldsymbol{k})|^m\rangle = c_m k^{-\zeta_m} \exp{[-k/k_d^{(m)}]}. \tag{2}$$

It was found [5, 6] that a good approximation could be obtained globally with

$$k_d^{(m)} = 2k_d^{(2)}/m, \qquad k_d^{(2)} = (13.5\eta)^{-1}, \tag{3}$$

with η being the Kolmogorov length scale. Eq. (3) means that for higher order velocity moments the crossover between the inertial range and the viscous range takes place at smaller value of k, i.e., the inertial range is less extended. As we want to examine the effects of the large scales, i.e., of the stirring subrange, we eliminate viscous effects by fitting (2) with fixed $k_d^{(m)}$ (according to (3)) *only* in the interval $[0, k_d^{(m)}]$. The resulting scaling exponents are (very slightly) corrected to guarantee $\zeta_3 = 1$, see [5, 6] for the procedure. $\zeta_3 = 1$ is required by Kolmogorov's structure equation [7], which strictly holds only in an isotropic situation. But as $\zeta_3 = 1$ is frequently enforced in the analysis of experimental data, we do it here as well. We represent the numerical results for the scaling exponents in terms of the deviations from the classical predictions $\zeta_m(Re) - m/3$. The results, which pertain now to both inertial range and stirring range, are shown in fig. 1 for four different Re. The deviations are decreasing with increasing Re.

Qualitatively this feature can be understood from the comparison with the *local* $\delta\zeta_m(k)$ for each Re, see ref. [5]. The $\delta\zeta_m(k)$ are calculated by *locally* employing the fit (2) to the spectra with $k_d^{(m)}$ fixed. The results are shown in fig. 2a for $Re = 1.05 \cdot 10^4$ and in fig. 2b for $Re = 1.4 \cdot 10^7$. Clearly, with increasing Re the inertial range extension exceeds more and more that of the stirring range, and thus the effect of the nonuniversal forcing diminishes.

To get quantitative information, we fitted the power law

$$\delta\zeta_m(Re) = c_m Re^{-\beta_m} \tag{4}$$

to the $\delta\zeta_m(Re)$ determined from the spectra. The results of the fit (4) for c_m and β_m are given in table 1. We find that β_m is close to 3/10 for all m.

A good way to understand this behavior is to consider the Euler part of the equations of motion using the representation by the Clebsch variables. In these variables the velocity field is written in terms of the two scalar functions $\lambda(\boldsymbol{x}, t)$ and $\mu(\boldsymbol{x}, t)$:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \lambda\boldsymbol{\nabla}\mu - \boldsymbol{\nabla}\phi, \tag{5}$$

where the potential $\phi(\boldsymbol{x},t)$ is determined from the incompressibility condition. Using these variables, the equation of motion

$$\partial_t \boldsymbol{u}(\boldsymbol{x},t) + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} = -\boldsymbol{\nabla} p(\boldsymbol{x},t) \tag{6}$$

reads

$$\partial_t \lambda(\boldsymbol{x},t) = \frac{\delta\mathcal{H}}{\delta\mu(\boldsymbol{x},t)}, \qquad \partial_t \mu(\boldsymbol{x},t) = -\frac{\delta\mathcal{H}}{\delta\lambda(\boldsymbol{x},t)}, \tag{7}$$

where the Hamiltonian $\mathcal{H}$ is defined as $\mathcal{H} = \int d\boldsymbol{x}|\boldsymbol{u}(\boldsymbol{x},t)|^2/2$. This formulation allows to introduce normal canonical variables $a(\boldsymbol{x},t)$ and $a^*(\boldsymbol{x},t)$, cf. [8],

$$\sqrt{2}a(\boldsymbol{x},t) = \lambda(\boldsymbol{x},t) + i\mu(\boldsymbol{x},t). \tag{8}$$

These canonical variables can be used to expose integrals of motion. In particular, the "occupation number" N and the "momentum" $\boldsymbol{P}$ of quasi particles,

$$N = \int |a_k|^2 d\boldsymbol{k}, \qquad \boldsymbol{P} = \int \boldsymbol{k}|a_k|^2 d\boldsymbol{k}, \tag{9}$$

are such integrals of motion [9, 10]. These integrals of motion could also be expressed as functions of the velocity field $\boldsymbol{u}(\boldsymbol{x},t)$, but are highly nonlocal in that representation. The main point is that the integral $\boldsymbol{P}$ can be non zero only in an anisotropic system, and therefore can lead us to scaling laws that capture the rate of decay of anisotropy as a function of $\boldsymbol{k}$. To proceed in this direction we first find by dimensional analysis that in an isotropic system the k-component of N, N_k, depends on k and the mean energy flux per unit time per unit mass, ϵ, according to

$$N_k\delta(\boldsymbol{k}-\boldsymbol{k}') = \langle a_k a^*_{k'}\rangle = C\epsilon^{1/3}k^{-13/3}\delta(\boldsymbol{k}-\boldsymbol{k}') \tag{10}$$

with C being a dimensionless constant. Of course, this result can be translated back to the standard Kolmogorov result for the energy $E_k = C\epsilon^{2/3}k^{-5/3}$. Next, we realize that in an anisotropic system this relation can change, since N_k can depend now on a dimensionless ratio of the "momentum flux" over the energy flux ϵ. The energy flux is defined by the equation

$$\partial_t\mathcal{H}_k + \frac{\partial\epsilon_k}{\partial k} = 0\,, \tag{11}$$

which in a stationary state requires ϵ_k to be $\boldsymbol{k}$-independent, $\epsilon_k = \epsilon$. The momentum flux $\boldsymbol{\pi}_k$, likewise, is defined by

$$-\,\partial_t\boldsymbol{P}_k = \boldsymbol{k}k^2\partial_t N_k(t) = \frac{\partial\boldsymbol{\pi}_k}{\partial k}. \tag{12}$$

In a stationary system $\boldsymbol{\pi}_k$ is also k-independent, and from eqs. (10) and (12) we see that the dimension of π can be written as $[\pi] = [\epsilon^{2/3}k^{1/3}] = length/(time)^2$. In an anisotropic system we seek a new solution for N_k,

$$N_k = C\epsilon^{1/3}k^{-13/3}f(\xi_k) \tag{13}$$

where the dimensionless parameter ξ_k is proportional to the ratio of the two fluxes π and ϵ. Since the two fluxes ϵ and π have different dimensionalities, the parameter ξ_k must involve k and ϵ to some powers. The unique combination that is proportional to π is

$$\xi_k = \frac{\pi}{\epsilon^{2/3} k^{1/3}} \ . \tag{14}$$

The function $f(\xi_k)$ is chosen such that for $\pi = 0$ Eq. (13) regains its Kolmogorov form, i.e. $f(0) = 1$. Asuming that for small π the correction to the spectrum is proportional to π, for small ξ we choose $f(\xi) = 1 + O(\xi)$. Notice that in (13) and (14) we assumed that the anisotropic correction to N_k is analytic in π. This is of course a crucial assumption, but it can be justified by an explicit calculation, and it was shown to be correct to all orders in perturbation theory, see [9]. Denoting the anisotropic correction as δN_k, we conclude that for small anisotropy, $\xi \ll 1$,

$$\delta N_k / N_k \propto k^{-1/3}. \tag{15}$$

Obviously, since $\boldsymbol{\pi}_k$ is odd in $\boldsymbol{k}$, cf. (9) or (12), so is also δN_k. In contrast, the double correlation function of the velocity field $F(\boldsymbol{k})$,

$$F(\boldsymbol{k}) \Delta(\boldsymbol{k} - \boldsymbol{k}') = Tr \langle \boldsymbol{u}(\boldsymbol{k}) \boldsymbol{u}^*(\boldsymbol{k}') \rangle \tag{16}$$

is even in $\boldsymbol{k}$. (Note that in accordance with our discrete simulations we define $F(\boldsymbol{k})$ with a Kronecker Δ.) Now, since $F(\boldsymbol{k})$ is even in $\boldsymbol{k}$, the lowest order correction due to anisotropy must be the second order. We can conclude therefore that

$$\delta F(\boldsymbol{k}) / F(\boldsymbol{k}) \propto k^{-2/3}. \tag{17}$$

To bring this result to a form that can be tested against the simulation, we multiply $F(\boldsymbol{k})$ by k (as numerically we of course use discrete wave vectors) and rewrite our result in the form

$$F(\boldsymbol{k}) = \langle |\boldsymbol{u}(\boldsymbol{k})|^2 \rangle = c_2 k^{-2/3} \left(1 + \alpha_m (kL)^{-2/3} \right) \exp\left[-k / k_d^{(2)} \right]. \tag{18}$$

(L is the outer length scale.) The same conclusion is obtained by considering the effect of external shear [11]. Since it introduces another time scale $\tau(k_L)$, spectral corrections should be $\propto \tau(k)/\tau(k_L) \propto k^{-2/3}$. On scales on which the shear dominates ($k \ll L^{-1}$) the second term in (18) will be even the leading contribution and thus will change the dominating k-power to $k^{-4/3}$. For a similar recent result concerning the $k^{-4/3}$-scaling law for small k in an anisotropic flow see [12]. The crossover from $k^{-4/3}$ to $k^{-2/3}$ can well be observed in experiments [12]. Here we restrict ourselves to $k > L^{-1}$. So the second term in (18) is an anisotropy correction, which will become smaller for increasing k due to isotropization by eddy decay. Yet it will render the scaling exponent k-dependent. Denoting the local scaling exponent $dln \langle |\boldsymbol{u}(\boldsymbol{k})|^2 \rangle / dlnk$ as $\zeta_2(k)$, we find from (18)

$$\zeta_2(k) = \frac{2}{3} + \frac{k}{k_d^{(2)}} + \frac{2}{3[1 + \alpha_2^{-1}(kL)^{2/3}]}. \tag{19}$$

As a function of k this expression has a minimum which is rather flat, and can lead to an apparent exponent $\zeta_2^{(app)}$. We estimate the value of this exponent by the minimum of (19) which approximately is

$$\zeta_2^{(app)} = \frac{2}{3} + \frac{10}{9}\left(\frac{9}{4k_d^{(2)}L}\right)^{2/5} |\alpha_2|^{3/5}. \tag{20}$$

Here, we used the fact that $1 \ll \alpha_2^{-1}(kL)^{2/3}$ near the minimum, justified a posteriori, as $k_d^{(2)}L \gg 9\alpha_2^{3/2}/4$. From (20), we can predict that the deviation of $\zeta_2^{(app)}$ from 2/3 goes down when Re increases, since $k_d^{(2)}L \propto Re^{3/4}$ [7]. We obtain

$$\delta\zeta_2^{(app)} \propto Re^{-3/10}. \tag{21}$$

Exactly the same calculation can be repeated for all other scaling exponents ζ_m, and all of them approach their Kolmogorov 41 values with the same scaling $\propto Re^{-3/10}$. The result of the calculation is

$$\zeta_m^{(app)} = \frac{m}{3} + sgn(\alpha_m)\frac{10}{9}\left(\frac{9}{4k_d^{(m)}L}\right)^{2/5} |\alpha_m|^{3/5}, \tag{22}$$

$$\delta\zeta_m^{(app)} = c_m Re^{-3/10} \tag{23}$$

with

$$\frac{c_m}{c_2} = sgn(\frac{\alpha_m}{\alpha_2})(\frac{m}{2})^{2/5}|\frac{\alpha_m}{\alpha_2}|^{3/5}. \tag{24}$$

From eq. (24) and table 1 we can determine the m-dependence of α_m. Assuming a power law dependence, we estimate $\alpha_m \propto m^{3.8\pm0.8}$, which is a rather strong m-dependence. This finding for α_m means that higher order moments are more affected by corrections to scaling than lower order ones. Together with the fact that their power law behavior is cut off at lower k vectors, cf. eq. (3), it means that it is harder to observe clean power laws in higher order moments for an appreciate range of scales.

In summary, the results of the numerical simulations of the Navier-Stokes equation with a reduced wave vector set approximation are in very close correspondence with the predictions of a theory in which the deviations from classical scaling is solely due to corrections to scaling. It is tempting to conjecture that if a full treatment of the Navier-Stokes dynamics results in genuine anomalous scaling that does not disappear in the limit of high Re, it has to do with the small scale structures that are underestimated in our simulations as well as in the analytical argument that refered to the Euler equations and neglected viscosity and boundary layers. Such a conjecture would be in accord with other theoretical expectations which link anomalous scaling to the creation of vortex structures like tubes and sheets.

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Tables

m	2	4	6	8	10
β_m	0.24	0.29	0.28	0.31	0.30
c_m	$+0.11$	-0.32	-1.30	-4.03	-7.02

Table 1

Fitparameters to eq. (4). If the m-dependence of c_m is expressed by a power law, we will get $c_m \propto m^{2.7\pm 0.5}$. The data are compatible with $c_m \propto (m-3)^{3\pm 0.5}$, indicating agreement with the Kolmogorov structure equation, which forbids corrections if $m = 3$. Here also the change of sign of the c_m is immediately grasped.

Figures

Figure 1: Results from our reduced wave vector set approximation for $|\delta\zeta_m(Re)|$ for m=2,4,6,8,10, bottom to top. The straight lines correspond to the scaling law $\delta\zeta_m(Re) \propto Re^{-3/10}$ as predicted by (23).

Figure 2: Scale resolved intermittency corrections $-\delta\zeta_m(k)$ for $m = 2, 4, 6, 8, 10$, bottom to top. In (a) we have $Re = 1.05 \cdot 10^4$, in (b) it is $Re = 1.4 \cdot 10^7$. The fit range is $[k/\sqrt{10}, k\sqrt{10}]$ for all k. For details see ref. [5, 6].

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